**MEASURE THEORY AND INTEGRATION – FINAL EXAM**
**Instructor:** Daniel Valesin

| | Q1 | Q2 | Q3 | Q4 | Q5 | Free points | $\sum$ |
|---|---|---|---|---|---|---|---|
| Total score: | 15 | 20 | 20 | 15 | 20 | 10 | 100 |
| Score obtained: | | | | | | 10 | |

1. Let $E \subset \mathbb{R}$ be a Borel measurable set with finite measure $m(E)$.
   (a) Show that there exists a Borel measurable set $A \subset E$ with $m(A) = m(E)/2$.
   (b) Show that for all $\varepsilon > 0$ there exists an open set $B$ with $E \subset B$ and $m(B) < m(E) + \varepsilon$.

2. (a) Let $f$ be a non-negative and measurable function defined on a measure space $(\Omega, \mathcal{A}, \mu)$. Show that

$$\mu(\{\omega : f(\omega) > \alpha\}) \leq \frac{1}{\alpha} \int_\Omega f \, d\mu \qquad \forall \alpha > 0.$$

   (b) Show that a measure space $(\Omega, \mathcal{A}, \mu)$ is $\sigma$-finite if and only if there exists an integrable function $f : \Omega \to \mathbb{R}$ so that $f(\omega) > 0$ for all $\omega$.

3. Assume that $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, are measurable functions with $f_1 \geq f_2 \geq \cdots \geq 0$ and such that $\int_\mathbb{R} f_n \, dm \to 0$ ($m$ denotes Lebesgue measure). Prove that $f_n \to 0$ almost everywhere.

4. Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be two measurable spaces.
   (a) Give the definition of the product $\sigma$-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$.
   (b) Show that, for every $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and every $\omega_1 \in \Omega_1$, we have $A_{\omega_1} \in \mathcal{A}_2$ (recall that $A_{\omega_1} = \{\omega_2 : (\omega_1, \omega_2) \in A\}$).

5. Let $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.
   (a) Assume $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers satisfying

$$\sum_{n=1}^{\infty} |a_n|^p < \infty, \qquad \sum_{n=1}^{\infty} |b_n|^q < \infty.$$

   Show that the series $\sum_{n=1}^{\infty} (a_n \cdot b_n)$ is convergent.
   *Hint.* Work on the measure space $(\mathbb{N}, P(\mathbb{N}), \mu)$, where $\mu$ is the counting measure. Note that a function $f : \mathbb{N} \to \overline{\mathbb{R}}$ can be associated to a sequence $(a_n)_{n\in\mathbb{N}}$ by setting $a_n = f(n)$. When is a function integrable, and what does integration mean in this space? What is $\mathcal{L}^p$?

(b) Again assume that $(a_n)_{n\in\mathbb{N}}$ satisfies $\sum_{n=1}^{\infty} |a_n|^p < \infty$. Show that

$$\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} = \sup\left\{\sum_{n=1}^{\infty}(a_n \cdot b_n) : (b_n)_{n\in\mathbb{N}} \text{ is a sequence satisfying } \sum_{n=1}^{\infty} |b_n|^q = 1\right\}.$$

*Hint.* To show that the left-hand side is less than or equal to the right-hand side, consider

$$b_n^\star = C \cdot \operatorname{sign}(a_n) \cdot (a_n)^{p-1},$$

where $C = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{-1/q}$.